INTEGRATION BY PARTIAL FRACTIONS Section - 4

The integrals of rational functions can be evaluated by splitting them into partial fractions. An expression containing a polynomial in numerator and another polynomial in denominator is called a rational function.

$\Rightarrow$ Rational Function $f(x) = \dfrac{\text{polynomial } N(x)}{\text{polynomial } D(x)}$

Case-I :

It the degree of numerator is less than the degree of denominator, we can split the rational functions into simpler fractions according to the factors of denominator.

For example :

Let $f(x) = \dfrac{x}{2x^2 - 5x + 3}$

$$\Rightarrow \quad f(x) = \frac{x}{(x-1)(2x-3)} = \frac{-1}{x-1} + \frac{3}{2x-3}$$

$\dfrac{-1}{x-1}$ and $\dfrac{3}{2x-3}$ are known as partial fractions of $f(x)$.

The integral of $f(x)$ can now be evaluated as the sum of the integrals of its partial fractions.

$$\Rightarrow \quad \int f(x)\, dx = -\log|x-1| + 3/2 \log|2x-3| + C$$

Case-II :

If the degree of $N(x)$ is greater than or equal to the degree of $D(x)$, we divide $N(x)$ by $D(x)$ so that the rational functional $\dfrac{N(x)}{D(x)}$ is expressed in the form $Q(x) + \dfrac{R(x)}{D(x)}$ where degree of $R(x)$ is less than the degree of $D(x)$.

Now as $Q(x)$ is a polynomial, it can be easily integrated and to integrate $\dfrac{R(x)}{D(x)}$ we make use of partial fractions as we have done in Case - I.

The resolution of $\dfrac{R(x)}{D(x)}$ into partial fractions depends upon the nature of the factors of denominator $D(x)$ as discussed below.

4.1 When denominator *g* (*x*) is expressible as the product of non-repeating linear factors

Let $D(x) = (x - a_1)(x - a_2) \ldots\ldots (x - a_n)$.

Then we assume that

$$\frac{R(x)}{D(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots\ldots + \frac{A_n}{x-a_n}$$

where $A_1, A_2, \ldots\ldots A_n$ are constants and can be determined by equating the numerator on RHS to the numerator on LHS or by substituting $x = a_1, x = a_2, \ldots\ldots, x = a_n$ respectively.

Illustration - 13 If $\int \frac{x^2 dx}{(x-1)(2x+3)} = \frac{x}{2} + A \log|x-1| + B \log|2x+3| + C$, then :

(A) $A = \frac{1}{5}, B = \frac{9}{20}$ **(B)** $A = \frac{-1}{5}, B = \frac{9}{20}$ **(C)** $A = \frac{1}{5}, B = \frac{-9}{20}$ **(D)** $A = \frac{-1}{5}, B = \frac{-9}{20}$

SOLUTION : (C)

Let $I = \int \frac{x^2\, dx}{(x-1)(2x+3)}$

The degree of numerator is not less than the degree of denominator. Hence we divide N by D.

$$\frac{x^2}{(x-1)(2x+3)} = \text{quotient} + \frac{\text{remainder}}{(x-1)(2x+3)}$$

$$= \frac{1}{2} + \frac{-\frac{1}{2}x + \frac{3}{2}}{(x-1)(2x+3)} = \frac{1}{2} + \frac{1}{2}\frac{3-x}{(x-1)(2x+3)}$$

We now split $\frac{3-x}{(x-1)(2x+3)}$ in two partial fractions.

Let $f(x) = \frac{3-x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$

where A and B are constants.

Equating the numerators on both sides :

$3 - x = A(2x+3) + B(x-1)$

Now there are two ways to calculate A & B.

1. Comparing the coefficients of like terms.
2. Substituting the appropriate values of x.

Method 1 :

Comparing the coefficients of x and x^0, we get :

$-1 = 2A + B$ and $3 = 3A - B$

On solving, we have $A = 2/5$ $B = -9/5$

Method 2 :

In $3 - x = A(2x+3) + B(x-1)$, put $x = 1, -3/2$

$x = 1 \Rightarrow 3 - 1 = 5A \Rightarrow A = 2/5$

$x = -3/2 \Rightarrow 3 + 3/2 = B(-3/2 - 1)$

$\Rightarrow B = -9/5$

Hence finally we have :

$$f(x) = \frac{\frac{2}{5}}{x-1} + \frac{-\frac{9}{5}}{2x+3}$$

$$\Rightarrow I = \int \left[\frac{1}{2} + \frac{1}{2} f(x)\right] dx$$

$$\Rightarrow I = \frac{x}{2} + \frac{1}{2}\int \frac{\frac{2}{5}}{x-1} dx + \frac{1}{2}\int \frac{-\frac{9}{5}}{2x+3} dx$$

$$\Rightarrow I = \frac{x}{2} + \frac{1}{5}\log|x-1| - \frac{9}{20}\log|2x+3| + C$$

Note : If the denominator contains only linear factors, the constants A and B can be calculated by the following method also.

$$f(x) = \frac{3-x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$$

A = Value obtained by substituting $x = 1$ in $f(x)$ after removing $(x-1)$ from denominator

$$\Rightarrow A = \left.\frac{3-x}{2x+3}\right]_{x=1} = \frac{3-1}{2+3} = \frac{2}{5}$$

B = Value obtained by substituting $x = -\frac{3}{2}$ in $f(x)$ after removing $(2x+3)$ from denomainator.

$$\Rightarrow B = \left.\frac{3-x}{x-1}\right]_{x=-\frac{3}{2}} = \frac{3+\frac{3}{2}}{\frac{-3}{2}-1} = -\frac{9}{5}$$

Illustration - 14 If $\int \frac{(x-1)dx}{(2x+1)(x+2)(x-3)} = A\log|2x+1| + B\log|x-2| + C\log|x-3| + D$, then :

(A) $A = \frac{-3}{35}, B = \frac{-1}{5}, C = \frac{2}{7}$ **(B)** $A = \frac{3}{35}, B = \frac{1}{5}, C = \frac{2}{7}$

(C) $A = \frac{-3}{35}, B = \frac{-1}{5}, C = \frac{-2}{7}$ **(D)** $A = \frac{3}{35}, B = \frac{1}{5}, C = \frac{-2}{7}$

SOLUTION : (A)

Let $f(x) = \dfrac{x-1}{(2x+1)(x-2)(x-3)}$

$= \dfrac{A}{2x+1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$

$\Rightarrow \quad A = \left.\dfrac{x-1}{(x-2)(x-3)}\right]_{x=-\frac{1}{2}} = -\dfrac{6}{35}$

$\Rightarrow \quad B = \left.\dfrac{x-1}{(2x+1)(x-3)}\right]_{x=2} = -\dfrac{1}{5}$

$\Rightarrow \quad C = \left.\dfrac{x-1}{(2x+1)(x-2)}\right]_{x=3} = \dfrac{2}{7}$

$\Rightarrow \quad \int f(x)dx = \dfrac{-6}{35}\int\dfrac{dx}{2x+1} - \dfrac{1}{5}\int\dfrac{dx}{x-2} + \dfrac{2}{7}\int\dfrac{dx}{x-3}$

$= -\dfrac{3}{35}\log|2x+1| - \dfrac{1}{5}\log|x-2| + \dfrac{2}{7}\log|x-3| + C$

4.2 When denominator $g(x)$ is expressible as the product of linear factors such that some of them are repeating

Let $\quad D(x) = (x-a)^k (x-a_1)(x-a_2) \ldots\ldots\ldots (x-a_n)$.

Then we assume that $\dfrac{R(x)}{D(x)} = \dfrac{A_1}{x-a} + \dfrac{A_2}{(x-a)^2} + \dfrac{A_3}{(x-a)^3} \ldots\ldots + \dfrac{A_k}{(x-a)^k} + \dfrac{B_1}{x-a_1} + \dfrac{B_2}{x-a_2} + \ldots\ldots\ldots \dfrac{B_n}{x-a_n}$

i.e. corresponding to the non-repeating factors we assume as in Section 4.1 and for each repeating factor of the type $(x-a)^k$, we assume partial fractions of the type :

$\dfrac{A_1}{x-a} + \dfrac{A_2}{(x-a)^2} + \dfrac{A_3}{(x-a)^3} \ldots\ldots\ldots + \dfrac{A_k}{(x-a)^k}$ where $A_1, A_2, \ldots\ldots A_k$ are constants.

Now to determine constants we equate numerators on both sides. Then from the resulting identity we can find all constants by substituting the various values of x as we have done in Section 4.1.

Illustration - 15 If $\displaystyle\int \frac{(\cos\theta+1)\sin\theta}{(\cos\theta-1)^2(\cos\theta-3)}d\theta = \log\left|\frac{\cos\theta-1}{\cos\theta-3}\right| + f(\theta) + C$, then $f(\theta) =$

(A) $\dfrac{1}{\sin\theta-1}$ **(B)** $-\dfrac{1}{\sin\theta-1}$ **(C)** $\dfrac{1}{\cos\theta-1}$ **(D)** $-\dfrac{1}{\cos\theta-1}$

SOLUTION : (D)

Let $\cos\theta = x \Rightarrow -\sin\theta\, d\theta = dx$

$\Rightarrow \quad I = -\displaystyle\int\frac{x+1}{(x-1)^2(x-3)}dx$

Let $f(x) = \dfrac{x+1}{(x-1)^2(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-3}$

Equating numerator on both sides,

$\Rightarrow \quad x+1 = A(x-1)(x-3) + B(x-3) + C(x-1)^2$

By taking $x = 1$, we get $B = -1$.

By taking $x = 3$, we get $C = 1$.

Comparing the coefficient of x^2, we get,

$0 = A + C \Rightarrow 0 = A + 1 \Rightarrow A = -1$

$\Rightarrow \quad I = -\int f(x)\,dx$

$= -\left\{\displaystyle\int\frac{-1}{x-1}dx + \int\frac{-1}{(x-1)^2}dx + \int\frac{1}{x-3}dx\right\}$

$\Rightarrow \quad I = \log|x-1| - \dfrac{1}{x-1} - \log|x-3| + C$

$\Rightarrow \quad I = \log\left|\dfrac{x-1}{x-3}\right| - \dfrac{1}{x-1} + C$

$\Rightarrow \quad I = \log\left|\dfrac{\cos\theta-1}{\cos\theta-3}\right| - \dfrac{1}{\cos\theta-1} + C$

4.3 When some of the factors of denominator D (x) are quadratic (which can not be factorised further) but not repeating.

Corresponding to each quadratic factor of the type $ax^2 + bx + c$, we assume partial fraction of the type $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar power of x in the numerator of both sides. (The constants can also be determined by using methods which we have used in Sections 4.1 and 4.2).

Illustration - 16

If $\displaystyle\int \frac{d}{x^3+1} = A\log\left|\frac{x+1}{\sqrt{x^2-x+1}}\right| + B\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$, then :

(A) $A = \dfrac{1}{\sqrt{3}}, B = \dfrac{1}{\sqrt{3}}$ (B) $A = \dfrac{1}{3}, B = \dfrac{1}{\sqrt{3}}$

(C) $A = \dfrac{1}{\sqrt{3}}, B = \dfrac{1}{3}$ (D) $A = \dfrac{1}{3}, B = \dfrac{1}{3}$

SOLUTION : (B)

Let $f(x) = \dfrac{1}{x^3+1} = \dfrac{1}{(x+1)(x^2-x+1)}$

$\Rightarrow \quad f(x) = \dfrac{1}{(x+1)(x^2-x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$

$\Rightarrow \quad 1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$

Comparing the coefficients of x^2, x, x^0.

$0 = A + B, \; 0 = -A + B + C, \qquad 1 = A + C$

$\Rightarrow \quad A = 1/3, \quad C = 2/3, \quad B = -1/3$

$\Rightarrow \quad f(x) = \dfrac{\frac{1}{3}}{x+1} + \dfrac{-\frac{x}{3}+\frac{2}{3}}{x^2-x+1}$

Let $I_1 = \dfrac{1}{3}\displaystyle\int \frac{dx}{x+1} = \frac{1}{3}\log|x+1| + C_1$

Let $I_2 = \displaystyle\int \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}dx = \frac{1}{3}\int \frac{2-x}{x^2-x+1}dx$

Express the numerator in terms of derivative of denominator.

$\Rightarrow \quad I_2 = -\dfrac{1}{6}\displaystyle\int \frac{2x-4}{x^2-x+1}dx$

$\Rightarrow \quad I_2 = -\dfrac{1}{6}\displaystyle\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$

$\Rightarrow \quad I_2 = -\dfrac{1}{6}\log|x^2-x+1| + \dfrac{1}{2}\displaystyle\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$

$\Rightarrow \quad I_2 = -\dfrac{1}{6}\log|x^2-x+1| + \dfrac{2}{2\sqrt{3}}\tan^{-1}\left(\dfrac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C_2$

$\Rightarrow \quad I_2 = -\dfrac{1}{6}\log|x^2-x+1| + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right) + C_2$

$\Rightarrow \quad \displaystyle\int \frac{dx}{x^3+1} = \int f(x)\,dx = I_1 + I_2$

$= \dfrac{1}{3}\log|x+1| - \dfrac{1}{6}\log|x^2-x+1| + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right) + C$

$= \dfrac{1}{3}\log\left|\dfrac{x+1}{\sqrt{x^2-x+1}}\right| + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right) + C$

4.4 When some of the factors of denominator D (x) are quadratic (which can not be factorised further) and repeating

For every quadratic factor of the type $(ax^2 + bx + c)$, we assume partial fractions of the type :

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \ldots\ldots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}$$

The constants are determined using method which we have already discussed in Sections 4.1 and 4.2.

4.5 When integrand contains only even powers of *x*

The integral in which the integrand contains only the even powers of x can be evaluated through the following steps :

Step – I : Consider integrand and put $x^2 = t$ in it.

Step – II : Make partial fractions of the resulting rational expression in t.

Step – III : Put $t = x^2$ again in the partial fraction and then integrate both sides.

Illustration - 17

If $\int \frac{x^2 dx}{x^4 - 1} = A \log \left| \frac{x-1}{x+1} \right| + B \tan^{-1} x + C$, then :

(A) $A = \frac{1}{2}, B = \frac{1}{2}$ **(B)** $A = \frac{1}{4}, B = \frac{1}{2}$ **(C)** $A = \frac{1}{2}, B = \frac{1}{4}$ **(D)** $A = \frac{1}{4}, B = \frac{1}{4}$

SOLUTION : (B)

$$\int \frac{x^2\, dx}{x^4 - 1} = \int \frac{x^2\, dx}{(x^2 - 1)(x^2 + 1)}$$

As the function contains terms of x^2 only, substitute $x^2 = t$ and then make partial fractions.

$$\frac{t}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1}$$

$\Rightarrow \quad t = A(t+1) + B(t-1)$

Put $t = \pm 1$ to get $A = 1/2,\ B = 1/2$.

$$\Rightarrow \quad \frac{t}{(t-1)(t+1)} = \frac{\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1}$$

Convert $t = x^2$ again before integrating.

$$\Rightarrow \quad I = \int \frac{x^2\, dx}{(x^2 - 1)(x^2 + 1)} = \int \frac{1/2}{x^2 - 1} dx + \int \frac{1/2}{x^2 + 1} dx$$

$$= \frac{1}{2} \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + \frac{1}{2} \tan^{-1} x + C$$